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A Functionalanalytic Characterization of Pure-Dimensional and Regular Stein Algebras

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0. INTRODUCTION

(0.1) In complex analysis of several variables it has turned out that Stein spaces are the equivalent to non-compact Riemann surfaces in classic function theory. On a Stein space there exist "enough" holomorphic functions, i.e., the global holomorphic functions describe completely the local function theory of this space. More precisely: let be given the algebra $\Gamma(X, \mathscr{O})$ of all holomorphic functions on a Stein space $(X, \mathscr{O})$, X the basic topological space and $\mathscr{O}$ the sheaf of germs of holomorphic functions on X; then both, X as well as $\mathscr{O}$, can be rediscovered by $\Gamma(X, \mathscr{O})$ uniquely up to biholomorphic maps. Namely, X turns out to be homeomorphic to the spectrum $\sigma\Gamma(X, \mathscr{O})$ and $\mathscr{O}$ can be obtained by a certain power series technique due to Forster [5].

We consider only reduced complex analytic spaces $(X, \mathscr{O})$. It is well known that $\Gamma(X, \mathscr{O})$ becomes a uniform Fréchet-nuclear algebra when endowed with the topology of compact convergence on X. A topological $\mathbb{C}$-algebra $\mathscr{A}$ is called a *Stein algebra* if there exists a Stein space $(X, \mathscr{O})$ such that $\mathscr{A}$ is topologically isomorphic to $\Gamma(X, \mathscr{O})$. In this paper we wish to characterize Stein algebras by "intrinsic properties," that is, by purely topologically algebraic properties which do not depend on the function theory of the corresponding Stein space. Thereby we achieve a reconstruction of holomorphy by functionalanalytic principles.

(0.2) The simplest class of non-trivial Stein algebras is the one of one-dimensional regular Stein algebras. Since they correspond with the class of non-compact Riemann surfaces they are called *Riemann algebras* in [11]. A first and rather difficult characterization of Riemann algebras is due to Richards [14]. A simple characterization has been gicen in [11] by using Gleason's famous theorem [7] and a theorem of Carpenter [3]. Unfortunately, these methods do not seem to be transferable to dimension $\geqslant 2$. In order to introduce analytic structure into spectra we apply a theorem

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independently proved by Basener and Sibony (see [1, 15]), instead of Gleason's theorem. Basener and Sibony developed a "hierarchy of Shilov boundaries" in order to generalize a deep theorem of Bishop and Wermer [17] on one-dimensional analytic structure (see (1.6)).

The main results of this paper are:

Theorem (5.2), *characterization of pure-dimensional Stein algebras*: The pure-dimensional Stein algebras are exactly the pure-dimensional strongly uniform Fréchet–Schwartz algebras with locally compact spectra.

Theorem (5.5), *characterization of regular Stein algebras*: The regular Stein algebras are exactly those strongly uniform Fréchet–Schwartz algebras with locally compact spectra such that all closed maximal ideals can be topologically generated by a fixed number of elements. (This number of generators is understood to be the minimal number, of course).

Recall that a uniform Fréchet algebra $\mathscr{A}$ is a *Schwartz algebra* if for any compact $K \subset \sigma\mathscr{A}$ there is a larger compact $L \subset \sigma\mathscr{A}$ such that the canonic restriction map $\mathscr{A}_L \to \mathscr{A}_K$ is a compact operator. In particular, every nuclear space is a Schwartz space.

A uniform Fréchet algebra is called *strongly uniform* if for every kernel ideal $\mathscr{E} \subset \mathscr{A}$, the algebra $\mathscr{A}/\mathscr{E}$ endowed with the canonic quotient topology is a uniform Fréchet algebra again. Our notion of dimension is the one of *Chevalley dimension* adapted for uniform Fréchet algebras (see (3.1)).

In Section 6 we give an example of a purely one-dimensional closed subalgebra of a Stein algebra which is not Stein. This algebra satisfies all conditions of Theorem (5.2) except one: there exists exactly one point φ_0 in its spectrum which possesses no compact neighborhood. Remarkably, the corresponding maximal ideal $\ker \varphi_0$ is principal. Our algebra is a countable dense inverse limit of purely one-dimensional Stein algebras.

(0.3) Now we outline the proof of the main theorems. In a preceeding paper [12] we proved for strongly uniform Fréchet–Schwartz algebras with locally compact spectra some principles: their spectra have a "good" topology, mappings $\sigma\mathscr{A} \to \mathbb{C}^n$ given by elements of $\mathscr{A}$ follow the principle of semicontinuity of fibre dimension, and a maximum modulus principle even hereditary for hulls. Notice that the two last-mentioned principles, coming from complex analysis, are proved there by means of the classic Weierstrass theorems; since such a "punctual" theory is not available in our setting, we had to develop new proofs.

A necessary condition for introducing analytic structure by Basener's theorem (1.6) is locally compactness of our spectra. In order to apply this theorem our algebras must be shown to satisfy two conditions, (1.6.1) and (1.6.2). This is the crucial step. It is managed by the above-mentioned theorems in [12] and some further theorems, namely, a maximum modulus principle (4.4) which combines the generalized Shilov boundaries and dimension, and the existence of special neighborhood bases (3.4). After that our algebra is recognized to be a pure-dimensional algebra of holomorphic functions. By a modification of a purely function theoretic theorem by Rossi (2.3) it is finally proved that this algebra is even Stein.

The characterization theorem for regular Stein algebras (5.5) is obtained mainly by (5.2). A non-trivial step is to verify the pure-dimensionality.

1. PRELIMINARIES

(1.1) A *Fréchet algebra* (=*(F)-algebra*) is a commutative locally convex, complete algebra over the complex field $\mathbb{C}$ with unit whose topology is generated by a countable number of seminorms.

Now let $\mathscr{A}$ be a (F)-algebra. By $\sigma\mathscr{A}$ we denote the *spectrum of* $\mathscr{A}$, the set of all continuous $\mathbb{C}$-algebra homomorphisms $\varphi\colon \mathscr{A} \to \mathbb{C}$ with $\varphi \neq 0$; as usual it is endowed with the Gelfand topology (=weak-*topology). $\mathscr{C}(\sigma\mathscr{A})$ is the algebra of all continuous functions on $\sigma\mathscr{A}$ endowed with the compact open topology.

The standard *Gelfand representation*

$$\Gamma\colon \mathscr{A} \to \mathscr{C}(\sigma\mathscr{A}), \qquad a \mapsto \hat{a},$$

given by setting $\hat{a}(\varphi) := \varphi(a)$ for $a \in \mathscr{A}$, $\varphi \in \sigma\mathscr{A}$, is a continuous $\mathbb{C}$-algebra homomorphism.

(1.2) Call $\mathscr{A}$ a *uniform Fréchet algebra* (=(uF)-algebra) if the Gelfand representation Γ induces a topological isomorphism of $\mathscr{A}$ onto a closed subalgebra $\Gamma(\mathscr{A}) \subset \mathscr{C}(\sigma\mathscr{A})$. For (uF)-algebras, thereby, *we shall identify* $\mathscr{A}$ *and* $\Gamma(\mathscr{A})$; *also we shall identify the elements* $f \in \mathscr{A}$ *of the algebra and their Gelfand transforms* $\hat{f} \in \Gamma(\mathscr{A})$. Mostly we shall consider (uF)-algebras whose spectra are assumed to be locally compact.

(1.3) Let X be a topological space. Then we call a countable *exhaustion*... $\subset K_n \subset K_{n+1} \subset$ ... of X by compact subsets an *admissible* one if for every compact subset $K \subset X$ there exists an index $n \in \mathbb{N}$ such that $K \subset K_n$. If X is assumed to be locally compact and if there exists an admissible exhaustion of X, then one can even chose an admissible exhaustion satisfying $K_n \subset \mathring{K}_{n+1}$, for all $n \in \mathbb{N}$.

Now let X be the spectrum of a (uF)-algebra $\mathscr{A}$. Then there exist admissible exhaustions, and every admissible exhaustion ... $\subset K_n \subset K_{n+1} \subset$ ... of X describes the topology of $\mathscr{A}$ by means of the corresponding seminorms $\|\cdot\|_{K_n}$, $n \in \mathbb{N}$. For $f \in \mathscr{A}$ and a compact set $K \subset X$ the seminorm $\|\cdot\|_K$ is defined as usual

$$\|f\|_K := \sup_{\varphi \in K} |f(\varphi)|.$$

Let $M \subset X$ be an arbitrary subset. By $\mathscr{A}_M$ we denote the separated completion of the restriction algebra $\{f|_M : f \in \mathscr{A}\}$ under the topology of uniform convergence on compact subsets of M. It is well known that

$$\sigma \mathscr{A}_M = \hat{M}_{\mathscr{A}},$$

where $\hat{M}_{\mathscr{A}}$ is the $\mathscr{A}$-convex hull of M in X; more precisely, $\hat{M}_{\mathscr{A}}$ is the union of all sets

$$\hat{K}_{\mathscr{A}} = \{\varphi \in X : |f(\varphi)| \leqslant \|f\|_K \text{ for all } f \in \mathscr{A}\}$$

with $K \subset M$ compact.

For compact M we have that $\mathscr{A}_M$ is a uniform Banach algebra with norm $\|\cdot\|_M$. But in general $\mathscr{A}_M$ need not even be a (uF)-algebra; namely, if M is not hemicompact then $\mathscr{A}_M$ is only a uniform locally-m-convex, complete algebra. For $M_1 \subset M_2 \subset X$ transitivity of localisation holds:

$$\mathscr{A}_{M_1} = (\mathscr{A}_{M_2})_{M_1}.$$

(1.4) Recall that a locally convex complete algebra $\mathscr{A}$ is a *Schwartz space* (=(S)-*space*) if for all Banach spaces $\mathscr{B}$, all continuous linear operators $\mathscr{A} \to \mathscr{B}$ are compact operators. For the theory of Schwartz spaces see Horvath's book [10, p. 271 ff.]. For (uF)-algebras the above condition can be reformulated more conveniently:

LEMMA. *Let $\mathscr{A}$ be a* (uF)*-algebra with spectrum X. Then $\mathscr{A}$ is a* (S)*-algebra if and only if for every compact subset $K \subset X$ there exists a* (*larger*) *compact subset $L \subset X$ such that the restriction map $\mathscr{A}_L \to \mathscr{A}_K$ is a compact operator.*

We omit the simple proof.

Note that any nuclear algebra is also a (S)-algebra.

(1.5) We shall use the terms "hull" and "kernel" analogically as in the theory of uniform algebras (cf. [6]).

Again, let $\mathscr{A}$ be a (uF)-algebra with spectrum X, and let $\mathscr{F} \subset \mathscr{A}$ be a set of functions on X. (In most cases $\mathscr{F}$ will be an ideal). Then the set

$$V(\mathscr{F}) := \{\varphi \in X : f(\varphi) = 0 \text{ for all } f \in \mathscr{F}\}$$

is called *the hull* (*in X*) *with respect to $\mathscr{F}$*. A set $M \subset X$ is called *a hull* if there exists a family $\mathscr{F} \subset \mathscr{A}$ of functions such that $M = V(\mathscr{F})$.

For a given subset $M \subset X$ we consider the ideal

$$k(M) := \{f \in \mathscr{A} : f|_M = 0\}.$$

It is called *the kernel (in $\mathscr{A}$) with respect to M*. An ideal $\mathscr{I} \subset \mathscr{A}$ is said to be a *kernel ideal* if it is the kernel with respect to $V(\mathscr{I})$.

Let $\mathscr{I} \subset \mathscr{A}$ be a closed ideal and $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ an admissible exhaustion of X. Then the quotient algebra $\mathscr{A}/\mathscr{I}$ carries the natural quotient topology given by the sequence of seminorms

$$\|f+\mathscr{I}\|_n := \inf_{g \in \mathscr{I}} \|f+g\|_{K_n}, \qquad n \in \mathbb{N}.$$

$\mathscr{A}/\mathscr{I}$ is a (F)-algebra under this topology. Now, assume moreover that $\mathscr{I}$ is a kernel ideal in $\mathscr{A}$. Then $\mathscr{A}/\mathscr{I}$ is a semisimple (F)-algebra; the uniform topology on $\mathscr{A}/\mathscr{I}$ given by the system of seminorms $\|f\|_{K_n \cap V(\mathscr{I})}$, $f \in \mathscr{A}$, $n \in \mathbb{N}$, is coarser than the above quotient topology. We often need coincidence of both topologies.

DEFINITION. A (uF)-algebra $\mathscr{A}$ is called *strongly uniform* (=(u*), for short) if for all kernel ideals $\mathscr{I} \subset \mathscr{A}$, the algebra $\mathscr{A}/\mathscr{I}$ endowed with the natural quotient topology is a (uF)-algebra again.

For some remarks on (u*F)-algebras see [12].

(1.6) Basener [1] and Sibony [15] independently introduced the concept of generalized Shilov boundaries.

Let $\mathscr{B}$ be a uniform Banach algebra. Denote by $\gamma\mathscr{B}$ the usual Shilov boundary. For $p \geqslant 1$ we introduce the *p-th Shilov preboundaries* $\tilde{\gamma}_p\mathscr{B}$:

$$\tilde{\gamma}_p\mathscr{B} := \bigcup_{(f_1,\ldots,f_p) \in \mathscr{B}^p} (\gamma\mathscr{B}_{\mathscr{V}(f_1,\ldots,f_p)}).$$

Here $\mathscr{V}(f_1,\ldots,f_p)$ is understood to be a subset of $\sigma\mathscr{B}$, in the natural way. Now define the *p-th Shilov boundaries* $\gamma_p\mathscr{B}$:

$$\gamma_p\mathscr{B} := \overline{\tilde{\gamma}_p\mathscr{B}}.$$

For $p=0$ set $\tilde{\gamma}_0\mathscr{B} := \gamma\mathscr{B}$. Then of course, $\gamma_0\mathscr{B} = \tilde{\gamma}_0\mathscr{B} = \gamma\mathscr{B}$. For an easy and useful exercise, compute $\gamma_p\mathscr{H}(P_n)$, $p \in \mathbb{N}$, where P_n is the closed n-polycylinder and

$$\mathscr{H}(P_n) = \{f \in \mathscr{C}(P_n) : f \mid \dot{P}_n \text{ holomorphic}\}.$$

If $(X, \mathscr{O})$ is a pure-dimensional complex analytic space and $K \subset X$ a compact subset then we have

$$\tilde{\gamma}_{n-1}(\Gamma(X,\mathscr{O}))_K = \gamma_{n-1}(\Gamma(X,\mathscr{O}))_K \subset \partial\hat{K},$$

∂ being the topological boundary. This assertion will be proved with much more generality in (4.4).

Now, we state Basener's theorem in a slightly modified form.

Theorem

HYPOTHESES. Let $\mathscr{B}$ be a uniform Banach algebra with spectrum K, and $f=(f_1,\ldots,f_n)\in\mathscr{B}^n$; let $L\subset K$ be a compact subset with

$$\gamma_{n-1}\mathscr{B}\subset L.$$

(1.6.1) Assume: $f(K-L)\cap(\mathbb{C}^n-f(L))\neq\varnothing$; denote by W the connected component of $\mathbb{C}^n-f(L)$ which meets $f(K-L)$. Moreover,

(1.6.2) let $\bar{f}^1(z)$ be a finite set, for all $z\in W$.

Then we have the

STATEMENT. *There exists a sheaf $\mathscr{O}$ of germs of continuous functions on $\bar{f}^1(W)$ such that $(\bar{f}^1(W),\mathscr{O})$ is a purely n-dimensional complex analytic space.*

Furthermore, the natural algebra monomorphism

$$\mathscr{B}_{\bar{f}^1(W)}\to\Gamma(\bar{f}^1(W),\mathscr{O})$$

is well defined; i.e., all elements of $\mathscr{B}$ are holomorphic functions on $\bar{f}^1(W)$.

Remark. Condition (1.6.2) can be viewed as a dimension condition (see Sect. 3). Under the hypotheses of this paper it will be the crucial step to fulfil conditions (1.6.1) and (1.6.2). In order to do so we had to develop the principle of semicontinuity of fibre dimension (3.2.2) and a hereditory maximum modulus principle (4.2) (for proofs see [12]) as well as further theorems in Sections 3 and 4.

2. SOME THEOREMS ON STEIN ALGEBRAS

(2.1) For the notion of *complex analytic spaces* see the book of Gunning and Rossi [8]. We consider only *reduced complex analytic spaces* $(X,\mathscr{O})$ such that X *has a coutable basis* for its topology. The algebra $\Gamma(X,\mathscr{O})$ of all holomorphic functions on $(X,\mathscr{O})$ becomes a uniform Fréchet nuclear algebra when endowed with the topology of compact convergence.

A (uF)-algebra $\mathscr{A}$ is called (reduced) *Stein algebra* if there exists a Stein space $(X,\mathscr{O})$ such that $\mathscr{A}$ is topologically isomorphic to $\Gamma(X,\mathscr{O})$. Call in particular, $\mathscr{A}$ a *regular Stein algebra* if the corresponding Stein space is even a Stein manifold. A given algebra of type $\Gamma(X,\mathscr{O})$ is Stein if and only if the natural evaluation map

$$j\colon X\to\sigma\Gamma(X,\mathscr{O})$$

is a homeomorphism (see [2, p. 140]).

The categories of Stein spaces and Stein algebras are antiequivalent. Hence the global functions on a Stein space describe the local function theory of this space completely. A great portion of the theory of Stein algebras is developped in Forster's papers [4, 5].

(2.2) In the following proposition we collect some properties of Stein algebras which appear in our characterization theorems in Section 5.

PROPOSITION. *Let* $\mathscr{A}$ *be a Stein algebra. Then* $\mathscr{A}$ *is a* (u*FS)*-algebra with locally compact*, (hemicompact), *and second countable spectrum* $\sigma\mathscr{A}$.

Proof. The (uF)-property of $\mathscr{A}$ and hence hemicompactness of $\sigma\mathscr{A}$ are well known. Locally compactness and the existence of a countable basis for $\sigma\mathscr{A}$ follow from the corresponding facts for X via the homeomorphism j.

The Schwartz property can be derived from Montel's theorem, or from the well known fact that $\Gamma(X, \mathscr{O})$ is even nuclear.

By Forster [5, p. 313], the quotient algebra $\mathscr{A}/\mathscr{I}$ is a Stein algebra for any closed ideal $\mathscr{I}$, and thus (uF) for kernel ideals $\mathscr{I}$. Hence Stein algebras are (u*F)-algebras.

(2.3) The following is a reformulation of Rossi's theorem, see [2, p. 144].

THEOREM. *Let* $\mathscr{A}$ *be a Stein algebra and* $\mathscr{A}_0 \subset \mathscr{A}$ *a closed subalgebra such that the adjoint spectral map*

$$p: \sigma\mathscr{A} \to \sigma\mathscr{A}_0$$

is a proper map. Then $\mathscr{A}_0$ *is a Stein algebra, too, whose spectrum is obtained by identifying those points in* $\sigma\mathscr{A}$ *which cannot be separated by* $\mathscr{A}_0$.

Proof. Assume $\mathscr{A} = \Gamma(X, \mathscr{O})$ with a Stein space $(X, \mathscr{O})$. Let $K \subset X$ be a compact set. Then

$$L := \widehat{p(K)}_{\mathscr{A}_0}$$

is compact. Since p is proper, $\bar{p}^1(L)$ is compact and hence $\hat{K}_{\mathscr{A}_0}$ as a closed subset of $\bar{p}^1(L)$ is also compact. Thus X is $\mathscr{A}_0$-convex.

Application of Rossi's theorem shows now that $\mathscr{A}_0$ is Stein. Moreover, it asserts the above stated structure of $\sigma\mathscr{A}_0$.

(2.4) A (uF)-algebra is Stein if it is "locally" Stein. More precisely:

PROPOSITION. *Let* $\mathscr{A}$ *be a* (uF)*-algebra with spectrum X. Then* $\mathscr{A}$ *is Stein if and only if*:

> *there exists a countable open covering* $(U_i)_{i \in I}$ *of* X *by* $\mathscr{A}$*-convex sets such that each* $\mathscr{A}_{U_i}$ *is Stein.*

Proof. If $\mathscr{A}$ is Stein then, trivially, the cover $\{X\}$ does the job.

Conversely, the given covering by Stein spaces U_i yields a countable base $\mathscr{U}$ for the topology of X consisting of Stein spaces, because any $\mathscr{A}_{U_i}$-convex set $V_i \subset U_i$ is holomorphically convex.

Thus the system $(\mathscr{A}_U)_{U \in \mathscr{U}}$ defines a presheaf of germs of holomorphic functions on X. Denote by $\tilde{\mathscr{A}}$ the associated sheaf (see [8, Chap. IV A]). Then $(X, \tilde{\mathscr{A}})$ is a complex analytic space.

This space is holomorphically separable since $\mathscr{A} \subset \Gamma(X, \tilde{\mathscr{A}})$, and it is holomorphically convex since X is $\mathscr{A}$-convex. Hence $(X, \tilde{\mathscr{A}})$ is a Stein space.

The pair $\mathscr{A} \subset \Gamma(X, \tilde{\mathscr{A}})$ satisfies the hypotheses of Theorem 2.3, for the map p is here the identity. Therefore $\mathscr{A}$ is a Stein algebra, as desired.

(In fact, by the Oka–Weil–Cartan theorem ([2, p. 145, Satz 10]) one could even prove $\mathscr{A} = \Gamma(X, \tilde{\mathscr{A}})$).

3. Chevalley Dimension and Some Neighborhood Lemmas

(3.1) We recall the notion of (complex) *Chevalley dimension for* (uF)-*algebras* which was introduced in [12]. For further theorems and remarks see Sections 4 and 5 in [12].

Let $\mathscr{A}$ be a (uF)-algebra.

For any $\varphi \in \sigma\mathscr{A}$ consider the integer $d(\varphi)$ defined as the minimum of all $n \in \mathbb{N}$ such that

> there exist $f_1, \ldots, f_n \in \ker \varphi$ and there exists a neighborhood U of φ such that the fibres of the mapping $(f_1, \ldots, f_n)$: $U \to \mathbb{C}^n$ are finite sets.

If this minimum does not exist set $d(\varphi) = \infty$. The *dimension of* φ *in* $\sigma\mathscr{A}$ is defined by

$$\dim_\varphi \sigma\mathscr{A} := \begin{cases} 0, & \text{if } \varphi \text{ is an isolated point in } \sigma\mathscr{A} \\ d(\varphi), & \text{otherwise.} \end{cases}$$

Let M be a subset of $\sigma\mathscr{A}$, e.g., a hull or open. Then define the *dimension of* φ *with respect to* M

$$\dim_\varphi M := \dim_\varphi \sigma\mathscr{A}_M.$$

For $M = \varnothing$ or $\varphi \notin \hat{M}_{\mathscr{A}}$ set $\dim_\varphi M = -1$.

It is well known that for Stein algebras the above dimension equals the topological Krull dimension (cf. [5]). We need the following two theorems which are proved in [12, Theorem 4.3, 5.2].

(3.2.1) THEOREM. *Let $\mathscr{A}$ be a* (u*FS)*-algebra and $Y \subset \sigma\mathscr{A}$ a compact hull. Then Y is a finite set.*

(3.2.2) PRINCIPLE OF SEMICONTINUITY OF FIBRE DIMENSION. *Let $\mathscr{A}$ be a* (u*FS)*-algebra with locally compact spectrum X, and let $f = (f_1, ..., f_n): X \to \mathbb{C}^n$ with $f_i \in \mathscr{A}$, $1 \leqslant i \leqslant n$.*

Then the map

$$X \to \mathbb{N} \cup \{0, \infty\}, \qquad \varphi \mapsto \dim_\varphi \bar{f}^1(f(\varphi))$$

is semicontinuous, i.e., for every $\varphi \in X$ there exists a neighborhood U of φ such that

$$\dim_\varphi \bar{f}^1(f(\varphi)) \geqslant \dim_\psi \bar{f}^1(f(\psi)), \qquad \textit{for all } \psi \in U.$$

(3.3) We also need the following application of 3.2.2. For a proof see 5.4 and 5.5 in [12].

THEOREM. *Let $\mathscr{A}$ be a* (u*FS)*-algebra with locally compact spectrum X such that* $\dim_\varphi X < \infty$, *for all $\varphi \in X$. Then the topology of X has a countable basis.*

(3.4) We have a *neighborhood lemma* which sharpens the one in [12, 3.1].

LEMMA. *Let $\mathscr{A}$ be a* (uF)*-algebra with locally compact spectrum X. Then every $\varphi \in X$ with* $\dim_\varphi X < \infty$ *prossesses a countable basis of neighborhoods consisting of $\mathscr{A}$-convex sets.*

Moreover, there is a special basis (3.4.1) which will be constructed in the proof below, and which we shall use in (5.1).

Proof. Let $\varphi \in X$ with $\dim_\varphi X = r$ be given. We may assume $r \geqslant 1$ since the case $r = 0$ is trivial. Thus there exist a compact neighborhood $K_0 \subset X$ of φ and functions $f_1, ..., f_r \in \ker \varphi$ such that the mapping

$$f = (f_1, ..., f_r): K_0 \to \mathbb{C}^r$$

has finite fibres. Assume $\bar{f}^1(0) = \{\varphi, \varphi_1, ..., \varphi_n\}$. By the local compactness of X there is another compact $\mathscr{A}$-convex neighborhood K of φ with

$$K \subset \mathring{K}_0 \qquad \text{and} \qquad \varphi_i \notin K, \qquad 1 \leqslant i \leqslant n.$$

Setting $\Delta_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$ we show:

(3.4.1) *The sets $V_\varepsilon := K \cap \bar{f}^1(\Delta_\varepsilon^r)$, $\varepsilon > 0$, form a $\mathscr{A}$-convex neighborhood basis for φ.*

Clearly, all V_ε are $\mathscr{A}$-convex neighborhoods of φ. But suppose the system

$(V_\varepsilon)_{\varepsilon>0}$ does not form a neighborhood basis for φ. Then there is a neighborhood W of φ with $V_\varepsilon \not\subset W$, for all $\varepsilon > 0$. Choose a $\psi_n \in V_{1/n} - W$ for each $n \in \mathbb{N}$; since $\psi_n \in K$, there exists a convergent subsequence ψ_{n_k} with $\lim \psi_{n_k} = \psi$. By $\bigcap_{n\in\mathbb{N}} V_{1/n} = \{\varphi\}$ we have $\psi = \varphi$.

Consequently, an infinite number of ψ_n belongs to W. That contradicts the construction of ψ_n. We are done.

(3.4.2) *Remark.* It follows from (3.4.1) that *there is an* $\varepsilon_0 > 0$ *such that*

$$V_\varepsilon \subset \mathring{K}, \qquad \textit{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0.$$

For the topological boundaries ∂V_ε *we have even*

$$\partial V_\varepsilon \subset K \cap (\bar{f}^1(\partial \Delta_\varepsilon^r)), \qquad 0 < \varepsilon \leqslant \varepsilon_0.$$

(3.5) Let $\mathscr{A}$ be a (uF)-algebra, $\varphi \in \sigma\mathscr{A}$ and $U \subset \sigma\mathscr{A}$ a neighborhood of φ. Evidently, the inequality

$$\dim_\varphi \sigma\mathscr{A} \geqslant \dim_\varphi \sigma\mathscr{A}_U$$

holds. We show that our standard hypotheses yield equality of these dimensions.

LEMMA. *Let* $\mathscr{A}$ *be a* (u*FS)-*algebra with locally compact spectrum. Then*

$$\dim_\varphi \sigma\mathscr{A} = \dim_\varphi \sigma\mathscr{A}_U$$

for all $\varphi \in \sigma\mathscr{A}$ *and all neighborhoods* $U \subset \sigma\mathscr{A}$ *of* φ.

Proof. We need only show "$\leqslant$." Without loss of generality we may assume:

$\dim_\varphi \sigma\mathscr{A}_U =: r$ with $1 \leqslant r < \infty$, and $U \subset \sigma\mathscr{A}$ a relatively compact neighborhood of φ. Since $(\ker\varphi)_U = \{h \in \mathscr{A}_U : h(\varphi) = 0\}$, there are $g_1,\dots, g_r \in (\ker\varphi)_U$ and a compact neighborhood $V \subset U$ of φ such that the mapping

$$(g_1,\dots, g_r)\colon V \to \mathbb{C}^r$$

has finite fibres. After eventually shrinking the neighborhood V one achieves

$$V \cap \bar{g}_1^1(0) \cap \cdots \cap \bar{g}_r^1(0) = \{\varphi\}.$$

Set $\delta := \min\{\|g_\rho\|_V : \rho = 1,\dots, r\}$; clearly $\delta > 0$. The restriction map $\ker\varphi \to (\ker\varphi)_U$ has dense range. Hence there are

$$f_1,\dots, f_r \in \ker\varphi \text{ such that } \|f_\rho - g_\rho\|_V < \delta/2, \qquad 1 \leqslant \rho \leqslant r.$$

Now, recall that by Shilov's idempotent theorem for (uF)-algebras, all relatively open connected components of a hull are as well hulls. Hence $V(f_1,\dots,f_r)\cap V$ is a hull which lies relatively compact in V.

From (3.2.1) one concludes that $V(f_1,\dots,f_r)\cap V$ consists only of a finite number of points. Now apply (3.2.2). Hence there exists another neighborhood $W\subset V$ of φ such that the mapping

$$(f_1,\dots,f_r)\colon W\to\mathbb{C}^r$$

has finite fibres. Thus

$$\dim_\varphi \sigma\mathscr{A}\leqslant r=\dim_\varphi \sigma\mathscr{A}_U.$$

4. Two Maximum Modulus Principles

(4.1) Definition. Let $\mathscr{A}$ be a (u*F)-algebra. A is called *maximum modulus algebra* if the equation

$$\|f\|_K=\|f\|_{\partial K}$$

holds, for all compact $K\subset\sigma\mathscr{A}$ without isolated points, and for all $f\in\mathscr{A}$. $\mathscr{A}$ is called a *hereditary maximum modulus algebra* if $\mathscr{A}/\mathscr{I}$ is a maximum modulus algebra for all kernel ideals $\mathscr{I}\subset\mathscr{A}$.

(4.2) Theorem (see [12, 6.3, 5.5]). *Let $\mathscr{A}$ be a* (u*FS)-*algebra with locally compact spectrum such that* $\dim_\varphi\sigma\mathscr{A}<\infty$ *for all* $\varphi\in\sigma\mathscr{A}$.

Then $\mathscr{A}$ is a hereditary maximum modulus algebra.

(4.3) For calculating the ρth Shilov boundaries $\gamma_\rho\mathscr{B}$, $\rho\geqslant 1$, it suffices to take a smaller index set than $\mathscr{B}^\rho$.

Lemma. *Let $\mathscr{B}$ be a uniform Banach algebra and $\mathscr{B}_0\subset\mathscr{B}$ be a dense sub-algebra. Then for $\rho\geqslant 1$, we have*

$$\gamma_\rho\mathscr{B}=\overline{\bigcup_{f\in\mathscr{B}_0^\rho}\gamma\mathscr{B}_{V(f)}}.$$

Proof. The inclusion "$\supset$" is trivial. For showing the inverse inclusion it suffices to prove

$$\tilde{\gamma}_\rho\mathscr{B}\subset\overline{\bigcup_{f\in\mathscr{B}_0^\rho}\gamma\mathscr{B}_{V(f)}}.$$

Let $\varphi \in \tilde{\gamma}_\rho \mathscr{B}$ be given. Then there exists $g = (g_1, ..., g_\rho) \in \mathscr{B}^\rho$ such that $\varphi \in \gamma \mathscr{B}_{V(g)}$. Since the Choquet boundary is a dense subset of the Shilov boundary we may even assume

$$\varphi \in \chi \mathscr{B}_{V(g)};$$

here χ denotes the Choquet boundary.

Thus for arbitrary $\varepsilon > 0$ and for every relative neighborhood $\tilde{U}$ of φ in $V(g)$ there exists a function $\tilde{h} \in \mathscr{B}_{V(g)}$ such that

$$\tilde{h}(\varphi) = \|\tilde{h}\|_{\tilde{U}} = 1 \qquad \text{and} \qquad \|\tilde{h}\|_{V(g) - \tilde{U}} < \varepsilon.$$

Now fix an arbitrary ε with $0 < \varepsilon < \frac{1}{3}$ and an $\tilde{U}$, and choose a function $\tilde{h}$ as above. $\tilde{h}$ can be approximated by a function $h \in \mathscr{B}$ such that

$$\|h - \tilde{h}\|_{V(g)} < \varepsilon, \|h\|_{\tilde{U}} \geqslant h(\varphi) = 1 \qquad \text{and} \qquad \|h\|_{V(g) - \tilde{U}} < 2\varepsilon.$$

Now choose

(i) a neighborhood $U \subset X =: \sigma\mathscr{B}$ of φ with

$$U \cap V(g) = \tilde{U},$$

(ii) a neighborhood $W \subset X$ of $V(g)$ such that

$$\|h\|_{W - U} < 3\varepsilon.$$

We may assume that, with sufficiently small $\delta > 0$, W has the shape

$$W_\delta = \{\varphi \in X : |g_i(\varphi)| < \delta, 1 \leqslant i \leqslant \rho\}.$$

For if not, then there exist $\varphi_n \in W_{1/n} - U$ with $|h(\varphi_n)| \geqslant 3\varepsilon$ for $n \in \mathbb{N}$ and, by compactness of X, a convergent subsequence of φ_n whose limit ψ belongs to $V(g) - U$ since

$$\bigcap_{n \in \mathbb{N}} W_{1/n} = V(g);$$

but that establishes the contradiction

$$3\varepsilon \leqslant |h(\psi)| < 2\varepsilon.$$

By hypothesis, $\mathscr{B}_0$ is a dense subalgebra of $\mathscr{B}_0$. Thus there exist $f_1, ..., f_\rho \in \ker \varphi \cap \mathscr{B}_0$ such that

$$\|f_i - g_i\|_X < \delta, \qquad 1 \leqslant i \leqslant \rho.$$

Because of $|g_i(\vartheta)| \leqslant |f_i(\vartheta) - g_i(\vartheta)| + |f_i(\vartheta)| < \delta$ we have $V(f) \subset W$.

Altogether we obtain

$$\|h\|_{U\cap V(f)} \geqslant 1$$

and

$$\|h\|_{V(f)-U} < 3\varepsilon < 1.$$

Hence $U \cap \gamma\mathscr{B}_{V(f)} \neq \varnothing$ (see [18, p. 62]). By our construction we obtain a net in $\bigcup_{f\in\mathscr{B}\mathfrak{g}} \gamma\mathscr{B}_{V(f)}$ if we run $\tilde{U}$ resp. U through a neighborhood basis of φ in $V(g)$ resp. X. This net converges to φ. Thus the inclusion "$\subset$" is proved.

(4.4) THEOREM. *Let $\mathscr{A}$ be a hereditary maximum modulus* (u*F)-*algebra with locally compact spectrum. Let U be a relatively compact open subset of $\sigma\mathscr{A}$ such that $\bar{U}$ is $\mathscr{A}$-convex.*
Suppose

$$s := \min_{\varphi\in U} \dim_\varphi U \geqslant 1.$$

$$\textit{Then } \gamma_{s-1}\mathscr{A}_{\bar{U}} \subset \partial U.$$

Proof. The case $s = 1$ is trivial by hypothesis of the maximum modulus principle, since U does not contain isolated points.

Now suppose $s > 1$. Since ∂U is compact it suffices by (4.3) to show

$$\bigcup_{f\in\mathscr{A}^{s-1}} \gamma(\mathscr{A}_{\bar{U}})_{V(f)} \subset \partial U.$$

Notice that the $\mathscr{A}$-convexity of $\bar{U}$ implies

$$(\mathscr{A}_{\bar{U}})_{V(f)} = \mathscr{A}_{\bar{U}\cap V(f)}!$$

Let $f \in (\ker\varphi)^{s-1} \subset \mathscr{A}^{s-1}$ be given such that $\varphi \in \gamma\mathscr{A}_{\bar{U}\cap V(f)}$. From the minimum dimension hypothesis we conclude

$$\min_{\psi\in\bar{U}\cap V(f)} \dim_\psi V(f) \geqslant 1.$$

Thus $\bar{U} \cap V(f)$ does not contain isolated points and hence $\mathscr{A}_{\bar{U}\cap V(f)}$ is a maximum modulus algebra. Therefore

$$\gamma\mathscr{A}_{\bar{U}\cap V(f)} \subset \partial(\bar{U} \cap V(f)) \subset \partial U,$$

as desired.

5. The Characterization Theorems

(5.1) First we prove a local version of the characterization Theorem (5.2). Its proof contains the crucial step for (5.2) and, partially, for the second main theorem of this paper, (5.5).

For a (uF)-algebra $\mathscr{A}$ consider the following open subset U of $\sigma\mathscr{A}$

$$U := \{\varphi \in \sigma\mathscr{A} : \varphi \text{ has a pure-dimensional neighborhood}\}.$$

THEOREM. *Let $\mathscr{A}$ be a* (u*FS)-*algebra with locally compact spectrum, and let $U \subset \sigma\mathscr{A}$ be as above.*

Then U can be given the structure of a holomorphically seperable complex analytic space such that $\mathscr{A}_U$ is an algebra of holomorphic functions on U which generates this analytic structure.

Proof. We must show that for each $\varphi \in U$ there is an $\mathscr{A}$-convex open neighborhood $V \subset U$ of φ such that $\mathscr{A}_V$ is a Stein algebra.

Let $\varphi \in U$ be given. The case $\dim_\varphi U = 0$ is trivial. Hence suppose $\dim_\varphi U = n > 0$.

Thus there exist $f_1, \dots, f_n \in \ker \varphi$ and a relatively compact neighborhood $V_1 \subset U$ of φ such that the map

$$f := (f_1, \dots, f_n) \colon V_1 \to \mathbb{C}^n$$

has finite fibres. The neighborhood lemma (3.4) and (3.4.1) yields an $\varepsilon > 0$ and a compact neighborhood $K \subset V_1$ of φ such that

$$V := \bar{f}^1(\Delta_\varepsilon^n) \cap K = \bigcap_{i=1}^{n} \bar{f}_i^1(\Delta_\varepsilon^n) \cap K$$

is a neighborhood of φ contained relatively compact in $\dot{K}$. By (3.4.2) we have

$$\partial V \subset \bar{f}^1(\partial \Delta_\varepsilon^n) \cap K.$$

The maximum modulus principles (4.2) and (4.4) give the inequality

$$\gamma_{n-1}\mathscr{A}_{\bar{V}} \subset \partial V,$$

since U is pure-dimensional and hence $\min_{\psi \in V} \dim_\psi V = n$.

Now, V has been constructed in such a way that $f(\varphi) \notin f(\partial V)$, and thus condition (1.6.1) is satisfied. Therefore we may apply Basener's theorem (1.6) setting $\bar{V} =: K$, $\partial V =: L$. Call W the connected component of $\mathbb{C}^n - f(\partial V)$ which contains $f(\varphi)$.

Thus there exists a sheaf $\mathscr{O}$ of germs of continuous functions defined on

$$V = (\bar{f}^1|_{\bar{V}})(W)$$

such that $(V, \mathcal{O})$ becomes an analytic space of pure dimension n with

$$\mathscr{A}_V \subset \Gamma(V, \mathcal{O}).$$

V is the intersection of $\mathscr{A}$-convex sets and hence $\mathscr{A}$-convex; by

$$\hat{M}_{\mathscr{A}_V} \supset \hat{M}_{\Gamma(V,\mathcal{O})}, \qquad \text{for} \quad M \subset V,$$

it is even holomorphically convex. $\Gamma(V, \mathcal{O})$ separates the points of V since $\mathscr{A}$ does so. Therefore $(V, \mathcal{O})$ is a Stein space with Stein algebra $\Gamma(V, \mathcal{O})$. By (2.3) we conclude that $\mathscr{A}_V$ is a Stein algebra, too.

Clearly, $\mathscr{A}_U$ generates the analytic structure in V since the restriction map

$$\mathscr{A}_U \to \mathscr{A}_V$$

has dense range. This completes the proof.

(5.2) CHARACTERIZATION THEOREM FOR PURE-DIMENSIONAL STEIN ALGEBRAS. *Let $\mathscr{A}$ be a pure-dimensional* (u*FS)*-algebra with locally compact spectrum. Then $\mathscr{A}$ is a pure-dimensional Stein algebra.*

Conversely, every pure-dimensional Stein algebra satisfies the above conditions.

Proof. The "Conversely" part of the proof is assertion (2.2).

Now let $\mathscr{A}$ be a pure-dimensional (u*FS)-algebra with locally compact spectrum X. Applying (5.1) we obtain that X can be given the structure of an analytic space such that $\mathscr{A}$ becomes an algebra of holomorphic functions. $\mathscr{A}$ separates the points of X. X is holomorphically convex since X is $\mathscr{A}$-convex. Hence X is even a Stein space.

$\mathscr{A}$ is a closed subalgebra of the algebra of all holomorphic functions on X, endowed with the analytic structure above. Since the corresponding spectral map is a homeomorphism we may apply Theorem (2.3) and obtain the desired fact that $\mathscr{A}$ is a pure-dimensional Stein algebra.

(5.3) *Remark.* The proofs for (5.1) and (5.2) yield a slightly stronger, but somewhat clumsy *characterization theorem*:

Let $\mathscr{A}$ be a pure-dimensional (u*F)*-algebra which is a hereditary maximum modulus algebra and whose spectrum is locally compact and second countable. Then $\mathscr{A}$ is a pure-dimensional Stein algebra.* Conversely, every pure-dimensional Stein algebra satisfies the above conditions.

(5.4) Now we wish to characterize the most important class of Stein algebras: *regular Stein algebras* (see (2.1)). The one-dimensional regular Stein algebras with connected spectrum—called *Riemann algebras*—were characterized in [11]. It seems unlikely that this characterization can be extended to the general case of regular Stein algebras by simply replacing the number of generators of the closed maximal ideals. A further special class of regular Stein algebras was characterized by Heal and Windham [9].

In the following theorem "*n-generated*" always refers to the minimal number of generators. In particular, a maximal ideal $\ker\varphi \subset \mathscr{A}$ is *locally topologically n-generated* if $(\ker\varphi)_K$ is in $\mathscr{A}_K$ topologically n-generated for all compact neighborhoods K of φ. Clearly, any topologically n-generated maximal ideal $\ker\varphi$ is locally topologically m-generated with an $m \leqslant n$.

Connectivity of the spectrum will be assumed only for technical simplicity; it is not essential for the proof of (5.5).

(5.5) CHARACTERIZATION THEOREM FOR REGULAR STEIN ALGEBRAS: *Let $\mathscr{A}$ be a* (u*FS)-*algebra with locally compact* (*and connected*) *spectrum X. Then the following assertions are equivalent*:

(i) *$\mathscr{A}$ is a regular Stein algebra*;

(ii) *for all $\varphi \in X$, the ideals* $\ker\varphi$ *are algebraically n-generated, with a fixed $n \in \mathbb{N}$*;

(iii) *for all $\varphi \in X$, the ideals* $\ker\varphi$ *are topologically n-generated, with a fixed $n \in \mathbb{N}$*;

(iv) *for all $\varphi \in X$, the ideals* $\ker\varphi$ *are locally topologically n-generated, with a fixed $n \in \mathbb{N}$.*

Proof. The necessity of all above conditions for regular Stein algebras is well known. In particular, the implication (i) $\Rightarrow$ (ii) follows from Forster [5, Satz 5.4]; for (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) use moreover Cartan's theorem which asserts that all finitely generated ideals in a Stein algebra are closed (see Forster [4, p. 314]).

For the converse implications it suffices to prove (iv) $\Rightarrow$ (i).

First we show that X is purely n-dimensional. By (3.2.2) and (3.5) we have

$$(+) \qquad \dim_\varphi X \leqslant n, \qquad \text{for all} \quad \varphi \in X.$$

Set $X_m := \{\varphi \in X\colon \dim_\varphi X \leqslant m\}$. In order to prove $\dim_\varphi X \geqslant n$ we shall show, step by step

$$X_0 = \varnothing, \ldots, X_{n-1} = \varnothing.$$

The case of $n = 0$ generators does not exist. For $n = 1$, the connectedness of X implies that either $X = X_0$ consists of a singleton or by (+), $X = X_1 - X_0$ is purely one-dimensional.

Now, let $n \geqslant 2$. Again, singletons cannot occur since X is connected, hence $X_0 = \varnothing$.

Suppose we have proved

$$X_i = \varnothing, \qquad \text{for} \quad i \leqslant m \leqslant n-2.$$

Then $X_{m+1} = X_{m+1} - X_m$ is an open and purely $(m+1)$-dimensional subset of X (if non-empty). By (5.1) X_{m+1} can be given the structure of a purely $(m+1)$-dimensional analytic space such that $\mathscr{A}_{X_{m+1}}$ becomes an algebra of holomorphic functions which generates this analytic structure. Then for any $\varphi \in X_{m+1}$ and any open, $\mathscr{A}$-convex neighborhood $U \subset X_{m+1}$ of φ we have:

$$\mathscr{A}_U = (\mathscr{A}_{X_{m+1}})_U \text{ is a Stein algebra.}$$

By Cartan's theorem (see [8]) the set of regular points of U lies open and dense in U. But for those points $\psi \in U$ the ideals $\ker \psi$ are locally topologically $(m+1)$-generated since $\mathscr{A}_U$ is Stein; this contradicts assumption (iv) because of $m+1 < n$.

Hence: $X_{m+1} = \varnothing$. Thus $X = X - X_{n-1}$ and by $(+)$, $X = X_n$.

By our characterization theorem (5.2) we obtain that $\mathscr{A}$ is a purely n-dimensional Stein algebra on the Stein space X. Since finitely generated ideals in Stein algebras are closed it follows from well known theorems of complex analysis, that the number of local topological generators of $\ker \varphi$ can be interpreted as the local embedding dimension of φ. Hence local embedding dimension and Chevalley dimension coincide for all $\varphi \in X$. Thus all $\varphi \in X$ are regular points and consequently, $\mathscr{A}$ is a regular Stein algebra.

6. A Non-Stein Algebra of Holomorphic Functions

(6.1) In this section we shall construct a *purely one-dimensional* (u*FS)-*algebra* $\mathscr{A}$ *which enjoys the following properties*:

$\mathscr{A}$ *is not Stein although*

$\mathscr{A}$ *is a closed subalgebra of a purely one-dimensional Stein algebra*;

there is a countable set $S \subset X = \sigma\mathscr{A}$ *such that*

(i) $\ker \varphi$ *is principal*, for all $\varphi \in X - S$,

(ii) $\ker \varphi$ *is two-generated*, for all $\varphi \in S$,

(iii) *there is exactly one point* $\varphi_0 \in X$ *which has no compact neighborhood in* X; $\varphi_0 \notin S$;

$\mathscr{A}$ *is a countable dense inverse limit of Stein algebras*;

$\mathscr{A}$ *can be represented as a fibre product algebra of three Stein algebras.*

(6.2) *Remark.* (i) $\mathscr{A}$ satisfies all conditions for Theorem (5.2) except the following one: there is exactly one point φ_0 without compact neighborhoods.

(ii) Our example also serves as a critical example for the theorem on characterizing Riemann algebras [11]: although $\ker \varphi_0$ is principal, there is no neigborhood of φ_0 equivalent to the unit disk.

(6.3) Set $X_n := \bigcup_{m=1}^{n} \{(z, w) \in \mathbb{C}^2 : (z - 1/m) \cdot w = 0\}$, for all $n \in \mathbb{N}$. *Then define*:

$$\mathscr{A} := \varprojlim_{n \in \mathbb{N}} (\mathscr{O}(\mathbb{C}^2))_{X_n}.$$

Obviously, X_n is an analytic subset of $\mathbb{C}^2$, $n \in \mathbb{N}$. Thus $\mathscr{A}$ is a countable dense inverse limit of Stein algebras.

As point sets we may identify the spectrum $\sigma\mathscr{A}$ and

$$X := \bigcup_{n=1}^{\infty} X_n;$$

$0 = (0, 0) \in X$ will correspond with the critical point $\varphi_0 \in \sigma\mathscr{A}$. It is easily seen that

$$X^* := X - \{0\}$$

is homeomorphic to $\sigma\mathscr{A} - \{\varphi_0\}$.

The natural restriction map yields a closed embedding of $\mathscr{A}$ into a Stein algebra.

$$\mathscr{A} \hookrightarrow \mathscr{O}(X^*)$$

Notice that $\mathscr{O}(X^*)$ is a Stein algebra since X^* is an analytic set in the domain of holomorphy $\mathbb{C}^2 - (\{0\} \times \mathbb{C})$. Thus we obtain a second representation of $\mathscr{A}$

$$\mathscr{A} = \left\{ \begin{array}{l} f \in \mathscr{O}(X^*) : f \text{ is bounded in } 0, \\ \quad \text{when restricted to } \mathbb{C}^* \times \{0\} \end{array} \right\},$$

where $\mathbb{C}^* := \mathbb{C} - \{0\}$.

Consequently, $\mathscr{A}$ is also a Schwartz space (even nuclear). For all compact sets $K \subset X$ with $0 \notin \hat{K}_{\mathscr{A}}$ the equation

$$\mathscr{A}_K = (\mathscr{O}(X^*))_K \text{ holds.}$$

Setting $S := \{(1/n, 0) \in \mathbb{C}^2 : n \in \mathbb{N}\}$ we conclude easily that

(i) $\ker \varphi$ is principal, for $\varphi \in X^* - S$,

(ii) $\ker \varphi$ is two-generated, for $\varphi \in S$.

ASSERTION. *For the critical point* φ_0 *we have*

$$\ker \varphi_0 = z \cdot \mathscr{A},$$

i.e., $\ker \varphi_0$ *is a principal ideal.*

Proof. Let $f \in \mathscr{A}$ be given with $f(0,0)=0$. Obviously, f/z restricted to X_n, $n \in \mathbb{N}$, belongs to $\mathscr{A}_{X_n}$. Since $\mathscr{A} = \varprojlim \mathscr{A}_{X_n}$, we have $f/z \in \mathscr{A}$. Thus

$$f = z \cdot (f/z) \in z \cdot \mathscr{A}.$$

ASSERTION. φ_0 *has in* X (*with respect to the* $\mathscr{A}$*-topology*) *no compact neighborhood.*

Proof. Set $K_n := X_n \cap \overline{\Delta_n^2}, \overline{\Delta_n^2}$ the dicylinder with radius $n \in \mathbb{N}$.

Then $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ is an admissible exhaustion of X. Suppose φ_0 has a compact neighborhood in X. Then there exists an index n_0 such that K_{n_0} is a compact neighborhood of φ_0.

Now consider the adjoint spectral map with respect to the embedding $\mathscr{A} \hookrightarrow \mathscr{O}(X^*)$

$$j \colon X^* \to X.$$

By the continuity of j the set

$$j^{-1}(\dot{K}_{n_0}) = X^* \cap \dot{K}_{n_0}$$

is open in X^* (with respect to the Euclidean topology) which is obviously wrong.

Contradiction!

Remark. Each sequence $\varphi_n \in X^* - (\mathbb{C} \times \{0\})$ which converges to $(0,0)$ with respect to the Euclidian topology on X, is divergent with respect to the $\mathscr{A}$-topology of X.

ASSERTION. $\mathscr{A}$ *is a* (u*)*-algebra.*

Proof. We need only show that $\mathscr{A}/\mathscr{I}$ is a (uF)-algebra for those kernel ideals $\mathscr{I}$ such that $(0,0) \in V(\mathscr{I})$, because $\mathscr{A}_K = (\mathscr{O}(X^*))_K$ for all compact

$$K \subset X^* \qquad \text{with} \quad 0 \notin \hat{K}_{\mathscr{A}}.$$

By the above remark one concludes that the critical point φ_0 is an isolated point even in hypersurfaces $V(f)$, $f \in \ker \varphi_0$, $f|_{\mathbb{C}\times\{0\}} \neq 0$. Hence we obtain the representation

$$\mathscr{A}/\mathscr{I} \cong \mathscr{A}/k(V(\mathscr{I}) - \{\varphi_0\}) \times \mathbb{C},$$

for all kernel ideals $\mathscr{J}$ with $\varphi_0 \in V(\mathscr{J})$. The right side algebra is a (uF)-algebra. Consequently, $\mathscr{A}$ is a (u*F)-algebra.

ASSERTION. *$\mathscr{A}$ is purely one-dimensional.*

Proof. The projection

$$p: X \to \mathbb{C}, \qquad p(z, w) = z - w$$

has locally finite fibres.

Finally, we give *a third representation* of $\mathscr{A}$ as a fibre product of three Stein algebras:

$$\begin{tikzcd} & \mathscr{A} \arrow[dl] \arrow[dr] & \\ \mathscr{O}(X^*) \arrow[dr] & & \mathscr{O}(\mathbb{C} \times \{0\}). \arrow[dl] \\ & \mathscr{O}(\mathbb{C}^* \times \{0\}) & \end{tikzcd}$$

All maps in this diagram are the natural restriction maps.

For the corresponding spectra there is the dual fibre coproduct diagram which enlightens once more the pathological $\mathscr{A}$-topology on X:

$$\begin{tikzcd} & X & \\ X^* \arrow[ur] & & \mathbb{C} \times \{0\}. \arrow[ul] \\ & \mathbb{C}^* \times \{0\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

All maps in this diagram are the natural inclusions.

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